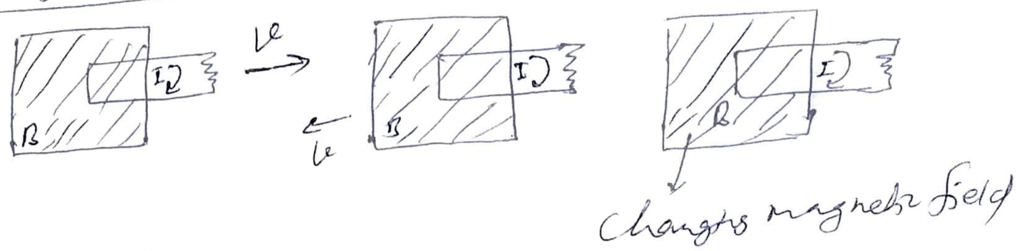
Electromagnetic Induction



$$\mathcal{E} = -\frac{d\phi}{dt}$$

A changing magnetic field induces an electric field.

$$\mathcal{E} = \oint E \cdot dl = -\frac{d\phi}{dt} \quad \text{---(1)}$$

$$\oint E \cdot dl = -\int \frac{\partial B}{\partial t} \cdot da \quad \text{---(2)}$$

↳ Faraday's Law in integral form

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$ Differential Form

in the static case ($B = 0$) $\oint E \cdot dl = 0$ or $\nabla \times E = 0$

⇒ Whenever the magnetic flux through a loop changes, an emf

$$\mathcal{E} = -\frac{d\phi}{dt} \quad \text{---(3)}$$

will appear in the loop.

Faraday's Law → electric fields induced by changing magnetic fields.

Lenz's Law → direction of flow of current

The Induced Electric Field

Faraday's discovery → Two distinct kinds of electric fields

1) those attributable directly to electric charges and 2) those associated with changing magnetic fields

① → Can be calculated using Coulomb's Law

② → Can be found by exploiting the analogy between Faraday's Law

Magnetic Vector Potential : The vector Potential

$\nabla \times E = 0 \longrightarrow E = -\nabla V$

So $\nabla \cdot B = 0 \longrightarrow$ introduction of a vector potential A in magnetostatics:

$$\boxed{B = \nabla \times A} \quad \text{--- (1)}$$

The potential formulation automatically takes care of $\nabla \cdot B = 0$ (since the divergence of a curl is always zero)

Ampere's Law

$\hookrightarrow \quad \nabla \times B = \nabla \times (\nabla \times A)$

$$= \nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 J \quad \text{--- (2)}$$

Electric potential $\rightarrow$ you can add to V any function whose gradient is zero (i.e. a constant) $\rightarrow$ without altering E.

Likewise $\rightarrow$ you can add to the magnetic potential any function whose curl vanishes (i.e. gradient of a scalar), with no effect on B.

We can exploit this freedom to eliminate the divergence of A:

$$\boxed{\nabla \cdot A = 0} \quad \text{--- (3)}$$

To prove :- Suppose that our original potential, A_0, is not divergenceless. If we add to it the gradient of λ ($A = A_0 + \nabla\lambda$), the new divergence is

$$\nabla \cdot A = \nabla \cdot A_0 + \nabla^2 \lambda$$

from eqn (3), if a function λ can be found that satisfies

$$\nabla^2 \lambda = -\nabla \cdot A_0$$

Mathematically, it is identical to Poisson's eqn

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

With $\nabla \cdot A_0$ in place of ρ/ϵ_0 as the "source".

We know how to solve Poisson's eq^n – "given the charge distribution, find the potential"

If $\rho \to$ zero at infinity

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\imath}\, d\tau',$$

Similarly if $\nabla \cdot A_0$ goes to zero at infinity, then

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot A_0}{\imath}\, d\tau'.$$

if $\nabla \cdot A_0 \to$ does not go to zero at infinity $\to$ we have to use other means to discover the appropriate λ.

But the essential point remains :– It is always possible to make the vector potential divergenceless

$B = \nabla \times A \to$ specifies the curl of A, but it doesn't say anything about the divergence, we are at liberty to pick that as we see fit $\to$ zero is the simplest choice.

With the condition on A, Ampere's law eq^n (2) becomes

$$\boxed{\nabla^2 A = -\mu_0 J} \quad \text{——(4)}$$

$\downarrow$

Poisson's eq^n.

Assuming J goes to zero at infinity, we can read off the sol^n.

$$\boxed{A(r) = \frac{\mu_0}{4\pi}\int \frac{J(r')}{\imath}\, d\tau'} \quad \text{——(5)}$$

For line and surface currents,

$$A = \frac{\mu_0}{4\pi}\int \frac{I}{r}\, dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{r}\, dl' ;$$

$$A = \frac{\mu_0}{4\pi}\int \frac{k}{r}\, da' \quad \text{--- (6)}$$

if the current does not go to zero at infinity →

we have to find other ways to get A.

A is not as useful as V. A → vector → fuss with components

It would be nice to get away with a scalar potential

$$B = -\nabla U \quad \text{--- (7)}$$

but incompatible with Ampere's law since the curl of a gradient is always zero.

Magnetostatic scalar potential → can be used if we stick to simply-connected, current free regions

→ of limited interest.

Vector potential → substantial theoretical importance.

Maxwell's Equations

Electrodynamics before Maxwell

(i) $\nabla \cdot E = \frac{1}{\varepsilon_0}\rho$ (Gauss's law)

(ii) $\nabla \cdot B = 0$ (no name)

(iii) $\nabla \times E = -\frac{\partial B}{\partial t}$ (Faraday's law)

(iv) $\nabla \times B = \mu_0 J$ (Ampere's law)

divergence of curl is always zero.

If you apply the divergence to number (iii)

$$\nabla \cdot (\nabla \times E) = \nabla \cdot \left(-\frac{\partial B}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \cdot B)$$

L.H.S zero → divergence of curl is zero.

R.H.S → by virtue of eqⁿ (ii)

If same thing applied to number (iv),

$$\underbrace{\nabla \cdot (\nabla \times B)}_{\text{zero}} = \underbrace{\mu_0 (\nabla \cdot J)}_{\text{not zero generally.}} \quad --(1)$$

→ for steady currents, $\nabla \cdot J = 0$ but when we go beyond magnetostatics → Ampere's Law cannot be right.

Ampere's Law is found to fail for nonsteady currents.

↓ Charging up a capacitor

How Maxwell fixed Ampere's Law

The problem is in the right side of eqⁿ (1), which should be zero, but isn't. Applying the continuity eqⁿ and Gauss's law

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot E) = -\nabla \cdot \left(\epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\Rightarrow \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \quad --(2)$$

Such modification changes nothing, as far as ~~magnetost~~ magnetostatics is concerned.

E → constant

Maxwell's term is hard to detect in ordinary electromagnetic experiments

$\downarrow$

It plays crucial role in the propagation of electromagnetic waves

Just as a changing magnetic field induces an electric field $\longrightarrow$ A changing electric field induces a magnetic field

Real confirmation of Maxwell's theory $\longrightarrow$ 1888 with Hertz experiment on electromagnetic waves.

Maxwell called the extra term

$\longrightarrow$ displacement current

$$J_d = \varepsilon_0 \frac{\partial E}{\partial t} \quad \text{— (3)}$$

$\downarrow$

? Is it current

misleading

displacement current $\longrightarrow$ charging capacitor

If the capacitor plates are very close together then the electric field between them is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0} \frac{Q}{A},$$

thus, between the plates

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A} \frac{dQ}{dt} = \frac{1}{\varepsilon_0 A} I$$

$\Rightarrow$ eq^n (2) in integral form

$$\oint B \cdot dl = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \left(\frac{\partial E}{\partial t}\right) \cdot da$$